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On Rudenko's Theorem For Discretely Valued Fields

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Abstract

S. Stein conjectured in 2000 that any dissection of a balanced plane polygon into triangles of equal area has an even number of triangles. Significant progress has been made but the conjecture remains open in general. Recently Rudenko proved Stein's conjecture for dissections with rational coordinates by proving a theorem about $\mathbb{Q}^2$-flows on trivalent graphs. We show the limits of Rudenko's theorem by establishing a necessary condition for it to hold for K^2-flows, where K is a valued field, and show that $\mathbb{R}$ fails to meet this condition for any choice of valuation. Moreover, we prove that this condition is sufficient when the valuation is discrete, proving Stein's conjecture for dissections with coordinates in a new class of number fields.

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1 Introduction

A polygon is called *balanced* if, given an orientation of its boundary, each sum over each equivalence class of parallel edge vectors is zero. Stein conjectured in 2000 that any dissection of a balanced polygon into triangles of equal area must use an even number of triangles [Ste00]. At first sight, this is a problem in Euclidean geometry, yet it admits an equivalent formulation in terms of $\mathbb{R}^2$-flows on cyclically oriented trivalent graphs. Rudenko, in a 2014 arXiv preprint [Rud25, v.1 from 2014], proved this equivalent formulation for $\mathbb{Q}^2$-flows by proving a stronger statement about $\mathbb{Q}^2$-flows.

Naturally, one asks: how far can Rudenko's theorem be strengthened? In this paper, we determine necessary and sufficient conditions for the statement of his theorem to extend to a discretely valued field, and give a necessary condition for his theorem to hold over an arbitrary valued field. In particular, we show that his theorem does not hold for $\mathbb{R}^2$-flows with arbitrary choice of valuation, and simultaneously determine a new class of number fields over which Stein's conjecture holds. Our work demonstrates that new ideas may be required to settle Stein's conjecture in its full generality.

Our notation and terminology will largely agree with the 2025 version of Rudenko's preprint [Rud25], also available on the arXiv. Consider a cyclically oriented trivalent (COT) graph G. For an abelian group A, an *A-flow* on G is a function $f\colon \vec{E}(G) \to A$ on the directed edges of G (we take each edge with both orientations) such that

1. For each pair of directed edges e, e' which share an undirected edge but are oriented in opposite directions, we have $f(e) + f(e') = 0$.

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2. For each vertex v and edges e_1, e_2, e_3 with source v, we have $f(e_1) + f(e_2) + f(e_3) = 0$.

Rudenko works with flows where $A = \mathbb{Q}^2$ or $A = \mathbb{Z}^2_{(2)}$. We will generalize this context. Let (K, ν) be a (non-archimedean) valued field, let R be its valuation ring, and let k be its residue field. In this setting, we consider K^2-flows on G, and call such a flow f *integral* if the image of f lies in R^2. Let $p\colon R^2 \to k^2$ be the projection; we call an edge e *even* if $p(f(e)) = 0$, and *odd* otherwise (we keep Rudenko's terminology, though ν may not be a 2-adic valuation). Consider the standard area form $\omega\colon K^2 \to K$ given by the determinant, fix a vertex v, and edges e_1, e_2 having source v with e_2 the immediate successor of e_1 in the cyclic ordering at v. We call $a(v) = \omega(f(e_1), f(e_2))$ the *area* of f at v, in analogy with dissections, and denote the *weight* of a vertex v of G by $m_\nu(v) = \nu(a(v))$. These quantities are both well-defined due to the flow condition. We write $m_\nu(G, f)$ for the minimal ν-weight among the vertices of G, and call a vertex *ν-minimal* if it attains this minimal weight. The dual formulation of Stein's conjecture leveraged by Rudenko is the following.

Conjecture 1 (Stein)**.** *Fix a COT graph G, let f be an $\mathbb{R}^2$-flow on G, and assume that all vertices of G with nonzero area at f have the same area. Then there is an even number of such vertices.*

It appears Rudenko was the first to consider this form of the conjecture, and it arises as follows. Given a dissection of a balanced polygon, we first insert degenerate triangles and vertices along the boundary so that (1) we may assume this is a triangulation and (2) we may assume the edges of the boundary are paired up so that, when the boundary is oriented, each pair of vectors sums to zero. The balancedness condition is precisely what is necessary for (2). The graph is obtained by dualizing this triangulation, the cyclic orientation obtained by ordering the edges of each triangle counterclockwise, and the flow obtained by taking the side vectors of each edge. Evidently, one may now replace $\mathbb{R}$ with any field K, and ask the same question about COT graphs, with no reference to the geometry of Stein's original conjecture. Rudenko showed the following, and used it to deduce the above for $K = \mathbb{Q}$. It is stated as Theorem 3.4. in [Rud25, v.1 from 2014] and as Theorem 1 in [Rud25].

Theorem 2 (Rudenko)**.** *Let $(K, \nu) = (\mathbb{Q}, v_2)$, where v_2 is the 2-adic valuation. Let G be a COT graph, and consider a K^2-flow on G. Then the number of ν-minimal vertices of G is even.*

For any valued field K, we show that having residue field $k = \mathbb{F}_2$ is a necessary condition for Rudenko's theorem to extend to K.

Proposition 3. *Let (K, ν) be a valued field with $k \neq \mathbb{F}_2$. Then there is a COT graph G and a K^2-flow on G with an odd number of ν-minimal vertices.*

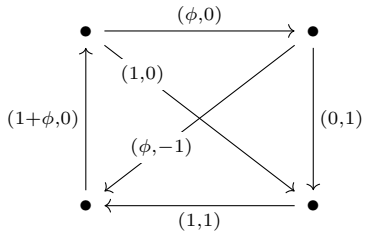
Under the additional assumption that the valuation of K is discrete, we show that this condition is also sufficient.

Theorem 4. *Let (K, ν) be a discretely valued field. Then the residue field of K is $\mathbb{F}_2$ if and only if for every COT graph G and K^2-flow on G the number of ν-minimal vertices of G is even.*

We begin with a proof of the proposition.

Proof of Proposition 3. Since $k \neq \mathbb{F}_2$, we may choose an element $\tilde{\phi} \notin \{0, 1\}$ of k, and a lift $\phi \in K$

of $\tilde{\phi}$. Consider the following K^2-flow on K_4:



The value of the flow on one orientation of each edge is given. The weight of the upper-left vertex is infinite, while the weight of every other vertex is 0. □

The obstruction to extending Rudenko's argument arises in the paragraph before his lemma 3 in [Rud25]. Here, he notes that for vertices with positive weight, the number of odd edges at the vertex is even. This uses a property of $k = \mathbb{F}_2$: whenever three vectors in k^2 are collinear, one of them must be the zero vector. In fact, $\mathbb{F}_2$ is the only field with this property.

Corollary 5. *For any non-archimedean valuation ν on $\mathbb{R}$ there is a COT graph G and $\mathbb{R}^2$-flow on G having an odd number of ν-minimal vertices.*

Proof. By Ostrowski's theorem, any such valuation would be equivalent to an extension of a p-adic valuation on $\overline{\mathbb{Q}}$, which has residue field $\overline{\mathbb{F}}_p$. □

The other direction of the proof follows Rudenko's argument, and we present it in the next section. Though the assumption that ν is discrete is necessary for the inductive proof we give, we conjecture that the statement of the theorem holds in general:

Conjecture 6. *Let (K, ν) be a valued field. Then the residue field of K is $\mathbb{F}_2$ if and only if for every COT graph G and K^2-flow on G the number of ν-minimal vertices of G is even.*

Furthermore, our theorem for discretely valued fields yields the corollary:

Corollary 7 (to Theorem 4)**.** *Let $P \subset \mathbb{R}^2$ be a balanced polygon and K a number field. If the minimal residue field degree over the prime ideals of K dividing the ideal (2) is 1, then every equidissection of P with vertices in K^2 has an even number of triangles.*

We refer the reader to Rudenko's derivation of his Theorem 2 from Theorem 1 in [Rud25], as this part of the argument proceeds identically. Notably, this corollary settles Stein's conjecture in the positive over a new (infinite) class of number fields, containing for instance the family $\mathbb{Q}(\sqrt[n]{2})$ for $n \geq 0$.

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1.2 AI Statement

No generative AI was used in the writing of this paper. The only time it was used in the course of this work was to clarify the meaning of some notation in Rudenko's paper [Rud25].

2 Sufficiency of $k = \mathbb{F}_2$

The remainder of this paper is dedicated to extending Rudenko's arguments to our setting to prove the remaining implication of Theorem 4. Suppose (K, ν) is a discretely valued field with residue field $k = \mathbb{F}_2$, let π be a uniformizer for ν, and let $\mathfrak{p} = (\pi)$ be the maximal ideal of R. Since $K = \mathrm{Frac}(R)$, and the conclusion of the theorem is invariant under scaling, we may assume that the flow is integral. We immediately make the following observation.

Lemma 8. *Let G be a COT graph with an integral flow f. Then the number of vertices with weight zero is even.*

Proof. Consider a vertex v with edges e_1, e_2, e_3 emanating from v. If $m(v) = 0$, the determinant $\omega(f(e), f(e'))$ is not in $\mathfrak{p}$, so its reduction modulo $\mathfrak{p}$ is nonzero. Thus $p(f(e))$ and $p(f(e'))$ are nonzero vectors. This is true of all pairs of edges, so all three are odd edges.

If $m(v) > 0$, the determinant $\omega(f(e_1), f(e_2))$ is in $\mathfrak{p}$, so $p(f(e_2))$ and $p(f(e_2))$ are linearly dependent; say $p(f(e_1)) = \lambda p(f(e_2))$ for some $\lambda \in k$. Further, we have that

$$p(f(e_3)) = -p(f(e_1)) - p(f(e_2)) = (\lambda - 1)p(f(e_2))$$

by the flow property. This is where we use the assumption that $k = \mathbb{F}_2$, as in this case we must have either $\lambda = 0$ or $\lambda = 1$. In the first case, $p(f(e_1)) = 0$, and in the second, $p(f(e_3)) = 0$. Thus there is at least one even edge emanating from v. If there were two, the third would be even as well, so v must have an even number of odd edges emanating from it.

The total number of edges in any trivalent graph is even; there are twice as many directed edges, to which vertices with positive weight contribute an even number of directed edges and vertices with weight zero contribute an odd number of directed edges. Hence there must be an even number of vertices of weight zero. □

The main idea of this proof is to exploit a connection to lattices over R. For a vertex v, we denote by $L(v)$ the R-sublattice of R^2 spanned by the values of f on the edges with source v. For an R-module M, let $\ell(M)$ denote its length.

Lemma 9. *For vertices v, v' with $L(v') \subseteq L(v)$, we have that $m(v') = m(v) + \ell(L(v)/L(v'))$. In particular, we have that $m(v) = \ell(R^2, L(v))$ for any vertex v.*

Proof. Let $L = L(v)$ and $L' = L(v')$, and $\ell = \ell(L/L')$. This quantity arises as $\chi(L/L') = \mathfrak{p}^\ell$, where $\chi(L/L')$ is an invariant defined in Serre [Ser13], I.§5. Let a be a K-linear map taking R^2 to L, and b a K-linear map taking L to L'. Then we have

$$m(v) = \nu(\det ab) = \nu(\det a) + \nu(\det b) = m(v') + \nu(\det b).$$

Now, if b is singular, so is ab, hence we have $\nu(\det b) = \infty$ and $m(v') = \infty$. Otherwise, by proposition 2 of [Ser13] III.§1, we have $\nu(\det b) = \ell$, completing the proof of the first sentence. For the second, observe that $R^2 = L(v)$ for any vertex v having weight 0. □

We may now prove the main theorem. By lemma 8, we may also assume the minimal weight is positive; pick such a counterexample (G, f) with minimal weight (using the fact that ν is a discrete valuation). As seen in the proof of lemma 8, the assumption that every vertex has positive weight implies that every vertex is incident to an even number of odd edges. This partitions G into a number of cycles in which all edges are odd; we call these *odd cycles*. We proceed by "eliminating" all odd cycles, to obtain a flow of minimal weight with all edges even, in which case we can divide the flow by the uniformizer π in order to obtain a counterexample of smaller weight.

Let $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ be an odd cycle, denote the oriented edge from v_i to v_{i+1} by $e_{i,i+1}$, and the reverse edge by $e_{i+1,i}$. Let e_i be the even edge with source v_i; this exists because v_i has precisely two odd edges. Denote the target of e_i by v_i'. We have that

$$\sum_{i=1}^{n} f(e_i) = \sum_{i=1}^{n} f(e_{i-1,i}) + f(e_{i+1,i}) = \sum_{i=1}^{n} f(e_{i,i+1}) + f(e_{i+1,i}) = 0.$$

Finally, denote the lattice $L(v_i)$ by L_i.

Lemma 10. *Let $v_1, v_2, \ldots v_n, v_{n+1} = v_1$ be an odd cycle in a COT graph with integral flow. Then there is some $i \in \{1, \ldots, n\}$ with $L_j \subseteq L_i$ for each $j \in \{1, \ldots, n\}$.*

Proof. We begin with an observation. If $u \in R^2$ is an odd vector, then the sublattices of R^2 containing u are linearly ordered by inclusion, as they are in inclusion-preserving correspondence with submodules of $R^2/\langle u \rangle \cong R$, which are precisely the ideals $\mathfrak{p} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3 \supset \cdots$ of R.

Without loss of generality, assume that L_1 is not strictly contained in any other lattice L_i, and assume there is a lattice it does not contain; we may choose the minimal j so that L_j is not contained in L_1. Certainly $L_{j-1} \subset L_1$, so the common odd vector $f(e_{j-1,j})$ of L_{j-1} and L_j is contained in L_1. By the above observation, we have either $L_j \subseteq L_{j-1}$ or $L_{j-1} \subseteq L_j$. The first would imply $L_j \subseteq L_1$. Since L_1 also contains $f(e_{j-1,j})$, the second would imply $L_{j-1} \subseteq L_1 \subsetneq L_j$, which is a contradiction by our choice of L_1. Hence there is no such j. □

Since every L_i is contained in L, lemma 9 says that those v_i with $L_i = L$ are precisely those vertices of the cycle with minimal weight. The following lemma states that there are an even number of such vertices.

Lemma 11. *Let L be the maximal lattice among the L_i as in the above lemma. Then the number of vertices for which $L_i = L$ is even.*

Proof. Consider a subgraph of G induced by the vertices $v_1, \ldots, v_k$ and $v_1', \ldots, v_k'$. This may be extended to a trivalent graph C by adding edges $e'_{i,i+1}$ for each i. We extend the flow f on G to a flow f_C on C by setting $f_C(e) = f(e)$ for each edge e not one of the $e'_{i,i+1}$, and $f_C(e'_{1,2}) = 0$, which uniquely determines the flow on the remaining edges.

Let L^0 be the lattice spanned by the even vectors in L. Since L is spanned by odd vertices, L^0 is a proper sublattice of L. Since we set $f(e'_{1,2}) = 0$, and the edges e_i are even, the vertices v_i' are incident to three even edges for all i. Hence $L(v_i') \subset L^0$ for all i. Thus the lattices at each of the vertices of C are contained in L. If $A \in \mathrm{GL}_2(K)$ is a linear map taking L to R^2, we have that the flow $f'_C = A \circ f_C$ is also integral. The vertices of C with weight zero for f'_C are precisely those vertices v whose lattices $L(v)$ (for f_C) equal L. We have shown that the only vertices whose lattices equal L are among the v_i, and lemma 8 yields that there are an even number of vertices with weight zero under f'_C. Thus there are an even number of i with $L_i = L$. □

We now describe how to create the counterexample of smaller weight. For each even oriented edge e, we set $f'(e) = f(e)$. Every odd edge is part of some odd cycle, say $v_1, v_2, \ldots, v_n, v_{n+1}$, with maximal lattice $L = L_1$. For each i, we set $f'(e_{i,i+1}) = f(e_{i,i+1}) - f(e_{1,2})$. This is even for all i because $f'(e_{1,2}) = 0$, so that v_1 now has two even edges, and thus three, and this iterates over the cycle. To avoid confusion, for each vertex v, let $L'(v)$ and $m'(v)$ denote the lattice and weight corresponding to v for the flow f'. Since the $f'(e_{i,i+1})$ are even, they are each contained in $L^0 \subsetneq L$, so that

$$m'(v_i) = m(v_1) + \ell(L/L'(v_i)) \geq m(v_1) + \ell(L/L^0) > m(v_1).$$

As such, we have not created any vertices with weight lower than the minimal weight among the vertices of the cycle, and by lemma 11 the number of vertices attaining this minimum has changed by an even number. As such, the minimal weight $m(G, f')$ is not any lower than $m(G, f)$, and the difference between the number of vertices with $m'(v) = m(G, f)$ and those with $m(v) = m(G, f)$ is even. Since we assumed there were an odd number of vertices with $m(v) = m(G, f)$, there is an odd (in particular nonzero) number of vertices with $m'(v) = m(G, f)$, and hence $m(G, f') = m(G, f)$. Since every edge is now even, the flow $\pi^{-1}f'$ is integral, and moreover $m(G, \pi^{-1}f') < m(G, f')$. This contradicts our minimal choice of f, completing the proof.

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